GRAVITATIONAL INSTANTONS AND BLACK PLANE SOLUTIONS IN 4-D STRING THEORY

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Abstract

We consider gauged Wess-Zumino models based on the non compact group $SU(2,1)$. It is shown that by vector gauging the maximal compact subgroup $U(2)$ the resulting backgrounds obey the gravity-dilaton one loop string vacuum equations of motion in four dimensional euclidean space. The torsionless solution is then interpreted as a pseudo-instanton of the $d = 4$ Liouville theory coupled to gravity. The presence of a traslational isometry in the model allows to get another string vacuum backgrounds by using target duality that we identify with those corresponding to the axial gauging. We also compute the exact backgrounds. Depending on the value of k, they may be interpreted as instantons connecting a highly singular big bang like universe with a static singular or regular black plane geometry.

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1 Introduction

Since Witten's discover [1] that singular solutions to the string vacuum equations of motion [2] can be represented by exact two dimensional conformal field theories known as gauged Wess-Zumino-Novikov-Witten models (GWZM) [3], a lot of work has been made in the last years about the subject, with special attention put on solutions of relevance in black hole physics and cosmology [4]. In $1+1$ dimensions the "famous" $SU(1,1)/U(1)$ coset representing a Schwarzchild like black hole has been exhaustively analyzed. Generalizations of this model as $SO(d-1,2)/SO(d-1,1)$ cosets were considered in [5,6], where a guess leading to the exact (to all orders in $1/k$) backgrounds was given.

Of course we are ultimately interested in realistic four dimensional models. Some of them, obtained essentially by taking tensor products of $SU(1,1)$'s and $U(1)$'s, were considered in [7,8]. A possible classification of cosets leading to effective target spaces with one time direction was given in [9].

In this paper we consider a model based on gauging the maximal compact subgroup $U(2)$ of the non compact group $SU(2,1)$. The interest is at least two-fold. First, the backgrounds by themselves represent a highly non trivial solution to the string equations, or matter coupled to gravity system; from general arguments the one loop solution should have euclidean signature and then represent some kind of gravitational instanton, but this view could be changed by considering the exact solution. Second, we think it is an instructive algebraic exercise to explicitely work out non abelian groups other than those related to the A_1 Lie algebra. The techniques, in particular the parametrizations, used here for $SU(2,1)$ are in principle extensive to general $U(p,q)$.

The paper is organized as follows. In Section 2 we set up general definitions and conventions, while Section 3 is devoted to the $SU(2,1)$ parametrizations. In Section 4 we describe the computation of the one loop effective backgrounds, in Section 5 the curvature and equations satisfied by them. In Section 6 we compute the (presumibly) exact backgrounds and conjecture possible interpretations. In Section 7 we quote the expressions of the one loop dual solution. Section 8 is devoted to the conclusions. An appendix divided in three sections is added, where we collect some useful formulae.

2 Conventions

A bosonic string that sweeps out an euclidean genus g world-sheet Σ embedded in a gravity-axion-dilaton d dimensional background on target space $\mathcal{M}$ is described by the action

$$S[X;G,B,D] = \frac{k}{4\pi}\int_\Sigma \left((G_{ab}(X) * + iB_{ab}(X))dX^a \wedge dX^b - \frac{1}{2k}D(X) * R^{(2)} \right) \tag{2.1}$$

where "*" stands for the Hodge mapping wrt some metric on Σ, $R^{(2)}$ being its Ricci scalar that satisfies $\int_\Sigma *R^{(2)} = 8\pi(1-g)$.

The Weyl invariance condition of this two dimensional sigma model imposes that, at one loop [1] the backgrounds satisfy the set of equations [2]

$$\begin{aligned} 0 &= R_{ab} - \nabla_a\nabla_b D - \frac{1}{4}H_{acd}H_b{}^{cd} \\ 0 &= -2\,\nabla_a\nabla^a D - \nabla_a D\nabla^a D + R - \frac{1}{12}H_{abc}H^{abc} + \Lambda \\ 0 &= \nabla^c(e^D H_{abc}) \end{aligned} \tag{2.2}$$

where $H \equiv dB$ and $\Lambda = \frac{26-d}{3}k$ (our definitions for curvature, ecc., are those of ref. [11]). These equations follow from the d-dimensional action on $\mathcal{M}$

$$I[G,B,D] = \int_\mathcal{M} e^D\,(*R + \nabla D \wedge *\nabla D - \frac{1}{12}H \wedge *H + *\Lambda) \tag{2.3}$$

A GWZM is defined as follows. Let G a Lie group, H a subgroup of G and $\mathcal{G},\mathcal{H}$ their respective Lie algebras. If $g:\Sigma \mapsto G$, $\omega(g) \equiv g^{-1}dg = -\overline{\omega}(g^{-1}) \in \mathcal{G}$ stand for the Maurer -Cartan forms, and $\mathcal{A} \in \mathcal{H}$ is a gauge connection, then the defining action of a GWZM is [3]

$$\begin{aligned} S[g,\mathcal{A}] &\equiv \frac{k}{4\pi}(I_{WZ}[g] + I_G[g,\mathcal{A}]) \equiv \frac{k}{4\pi}(I_0[g] + i\Gamma[g] + I_G[g,\mathcal{A}]) \\ I_{WZ}[g] &= \frac{1}{2}\int_\Sigma tr(\omega(g)\wedge *\omega(g)) + \frac{i}{3}\int_{\mathcal{B},\partial\mathcal{B}=\Sigma} tr(\omega(g)\wedge\omega(g)\wedge\omega(g)) \\ I_G[g,\mathcal{A}] &= \int_\Sigma tr(\,-\mathcal{A}\wedge(*+i1)\,\omega(g) \;+\; \mathcal{A}\wedge(*-i1)\,\overline{\omega}(g) \\ &- \; g\,\mathcal{A}\,g^{-1}\wedge(*+i1)\mathcal{A} \;+\; \mathcal{A}\wedge *\mathcal{A}) \end{aligned} \tag{2.4}$$

where "tr" is normalized in such a way that the lenght of a long rooth of $\mathcal{G}$ is 2 [12]. This action is invariant under the gauge transformations [2]

[1] Strickly speaking, at first order in $\frac{1}{k} \equiv \alpha'$, see i.e. [10].

[2] A slightly modified version of the action (2.4) and gauge transformations (2.5), the so called "axial" gauging, is possible if $\mathcal{H}$ contains abelian subalgebras; the effective target is different but both theories are equivalent (dual), in agreement with current algebra arguments.

$$\begin{aligned} g^h &= h\,g\,h^{-1} \\ \mathcal{A}^h &= h\,\mathcal{A}\,h^{-1} - \overline{\omega}(h) \end{aligned} \qquad (2.5)$$

for an arbitrary map $h : \Sigma \mapsto H$.
If we pick a basis $\{T_a,\, a = 1, \ldots, \dim H\}$ in $\mathcal{H}$, then by integrating out the gauge fields in I_G we obtain the one loop order effective action

$$\begin{aligned} S_{eff}[g] &= \frac{k}{4\pi} W[g] - \frac{1}{8\pi} \int_\Sigma D(g) * R^{(2)} \\ W[g] &= I_{WZ}[g] + \tilde{I}[g] \\ \tilde{I}[g] &= -2 \int_\Sigma \frac{1}{l} (\lambda^c)^{ab}\, a_a \wedge (* - i1) b_b \end{aligned} \qquad (2.6)$$

where $l = l(g)$ and $\lambda^c = \lambda^c(g)$ are the determinant and the cofactor matrix of

$$\lambda_{ab}(g) = \frac{1}{2}\, tr(T_a T_b - g T_a g^{-1} T_b) \qquad (2.7)$$

and

$$\begin{aligned} i\,2\,a_a &= tr(T_a\,\omega(g)) \\ i\,2\,b_a &= tr(T_a\,\overline{\omega}(g)) \end{aligned} \qquad (2.8)$$

Clearly the gauge invariance condition $S_{eff}[g^h] = S_{eff}[g]$ makes the effective target dependent on $d = \dim G - \dim H$ gauge invariant field variables constructed from g. The d dimensional metric and torsion are read from $W[g]$. The dilaton field

$$D(g) = \ln |l(g)| \qquad (2.9)$$

comes from the determinant in the gaussian integration leading to (2.6) after convenient regularization [9]. [3]

It is undoubtly of major importance to get $d = 4$ target spaces since they can represent realistic backgrounds for string theory, with implications in cosmology and black hole physics in particular. Models with one time direction have been clasified in [9]. Most of them consist of groups product of $SU(1,1)'s$ and $U(1)'s$ (see however [5], where the only "less" trivial SO(3,1)/SO(2,1) coset is briefly considered). Unfortunately one of the most interesting targets, the "stringy" Schwarzchild solution (and more generically, geometries with a high degree of isometries), has evaded us. A naive explanation of this fact could be the following one: since at one loop $R_{ab} = 0$ and $D = const.$ for this solution, we would have to have (up to g-independent normalizations) $l(g) = 1$. But from (2.7) we see that

[3]Because λ transforms as a 2-tensor in the adjoint representation of H, $D(g)$ will be gauge invariant for subgroups with semisimple Lie algebra; for non semisimple subalgebras action (2.6) does not exist, see below.

the λ matrix is null when we approach to $g = 1$, and certainly it cannot have a non-vanishing determinant. More generally speaking, if G is semisimple we can always choose an orthogonal set of generators in $\mathcal{G}$ of non-zero norm; if we write $g = TU$ with $U \in H$ and $T \in G/H$ then from (2.7) we get

$$\lambda(g) = (1 - S(T)R(U))^t h \tag{2.10}$$

where $R(U)$ is the adjoint representation matrix of U, $S(T)$ contains the adjoint action of the coset element T on the $\mathcal{H}$ generators and h is the Killing-Cartan metric on $\mathcal{H}$. For elements in $H(S(1) = 1)$, λ becomes singular on some submanifold (the target space nature of it to be elucidated) and $l(g) \neq 1$. If G is not semisimple, then the Killing-Cartan form has null eigenvalues and λ does not exists in general. In any case is hard to see how a singular target space (and, to one loop at least, it should be!) could raise with a constant dilaton in the present context of GWZM. Maybe the non abelian duality transformations recently introduced [13,14] could indirectly lead to an exact conformal field theory representation of the stringy (and others) Schwarzchild black hole. [4]

[4] We point out that the S-duality [15] recently introduced in the context of superstring theory does not hold here; in fact our solutions has non zero cosmological constant.

3 The SU(2,1)/U(2) model

Coming back to our problem, a certainly non trivial four dimensional target we would get by considering $G = SU(2,1)$ and $H = U(2)$. From general arguments it will have (at one loop!) euclidean signature [9], and so it could represent some kind of "gravitational instanton" in the general sense of reference [16]. [5] So let us concentrate on this model. In view of the gauge invariance of the theory, it will be of most importance to fix a convenient parametrization. We will denote vectors with bold-type letters; matrices will be understood from the context.

An arbitrary element $g \in SU(2,1)$ admits the coset decomposition wrt its maximal compact subgroup $U(2)$,

$$g = T(\mathbf{c})H(U, u^*) \tag{3.1}$$

where T, H are given in eqns. (A.4). Clearly the SU(2,1) topology is $\Re^4 \times S^3 \times S^1$. Now, outside the origin of $\mathbb{C}^2$ the complex 2-vector $\mathbf{c}$ can be uniquely written as $\mathbf{c} = s\,\mathbf{n}$, with $s \equiv (\mathbf{c}^\dagger \mathbf{c})^{\frac{1}{2}}$ being the radial coordinate of $\Re^4$ and $\mathbf{n}^\dagger \mathbf{n} = 1$. The unitary vector $\mathbf{n}$ is in one-to-one correspondence with a $SU(2)$ matrix

$$\mathbf{n} \equiv \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \Leftrightarrow N \equiv \begin{pmatrix} n_1^* & n_2^* \\ -n_2 & n_1 \end{pmatrix} \tag{3.2}$$

Since an arbitrary element of $U(2)$ can be written as

$$U = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} P \tag{3.3}$$

with $P \in SU(2)$ and $u \equiv e^{i\varphi} = detU$, we can parametrize the $U \in U(2)$ in (3.1) as

$$U = N^\dagger \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} P\, N \tag{3.4}$$

and then we rewrite g in the form [6]

$$g = H(N^\dagger, 1)\, e^{t\lambda_4}\, e^{i\frac{\varphi}{2}(\lambda_3 + \sqrt{3}\lambda_8)} H(P, 1)\, H(N, 1) \tag{3.5}$$

where the relations $N\,\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and (A.6) were used. Finally, if according to (C.3) we introduce

[5] By means of a Wick rotation we can get $(++--)$ signature; it corresponds to gauging the $U(1,1)$ subgroup.

[6] From now on we will use the variable $t \in [0, \infty)$ and the symbols $s \equiv \sinh t$, $c \equiv \cosh t$.

$$
\begin{array}{rcl}
X & \equiv & e^{i\frac{\varphi}{2}\sigma_3}\, P = e^{i\frac{\theta}{2}\sigma_3}\, \overline{X}\, e^{-i\frac{\theta}{2}\sigma_3} \\
V & \equiv & e^{i\frac{\theta}{2}(1-\sigma_3)} N \quad \in U(2)
\end{array}
\tag{3.6}
$$

we obtain

$$
g = H(V^\dagger, 1)\, e^{t\lambda_4}\, e^{i\frac{\sqrt{3}}{2}\varphi\lambda_8}\, H(\overline{X}, 1) H(V, 1)
\tag{3.7}
$$

It is clear from this parametrization that V is a gauge variable and decouple from the model. The remaining four gauge invariant variables (for example, (t, φ, x_0, x_3)) locally parametrize the effective target manifold whose topology might be naively identify with $\Re^2 \times \mathcal{D}$ where $\mathcal{D}$ is a disk. This can be seen from the fact that according to (3.1,4) and (A.6), the (complex) variables

$$
\begin{array}{rcl}
\mathbf{c}^\dagger\, N^\dagger\, U\, N\, \mathbf{c} & = & s^2\,(x_0 + i x_3)\, e^{i\frac{\varphi}{2}} = s^2\,(p_0 + i p_3) u \\
trU & = & 2\, x_0\, e^{i\frac{\varphi}{2}} = p_0(1+u) - i\,(1-u)\, p_3
\end{array}
\tag{3.8}
$$

are the gauge invariant ones, and belongs to $\Re^2$ and $\mathcal{D}$ respectively. [7] However as follows from (2.10), the origin of $\Re^2$ as well as the boundary of the disk will become singular.

We remark that X belongs to $SU(2)$ only "locally" , but not globally as P does; it rises from parametrizing a $U(2)$ matrix as a $SU(2) \times U(1)$ element in (3.3), $U = e^{i\frac{\varphi}{2}} X$. It is useful to carry out computations and we will also consider it in what follows, as well as with

$$
\begin{array}{rcl}
V & = & e^{i\frac{\phi}{2}}\,(v_0 1 + i\, \mathbf{v}\cdot\sigma) \\
1 & = & {v_0}^2 + \mathbf{v}\cdot\mathbf{v}
\end{array}
\tag{3.9}
$$

in Section 6.

[7] The complex variable trU (that encodes $det\, U = u$) is the gauge invariant variable describing the coset $U(2)/Adj\, U(2) \equiv \mathcal{D}$. We thank M. Blau for a discussion on this point.

4 Computation of the one loop metric

In this section we will describe with some detail the calculations of the one loop backgrounds. The parametrization (3.7) (with $V=1$) will be assumed.

First of all, we have to choose a convenient basis in $\mathcal{H}$. We take the following generators ($(\check{e}_i)_j = \delta_{ij}$)

$$\begin{aligned} T_i &= \lambda_i - \frac{1}{\sqrt{3}}\,\lambda_8\,\delta_{i,3}\,, \quad i=1,2,3 \\ T_4 &= -\frac{2}{\sqrt{3}}\,\lambda_8 \end{aligned} \tag{4.1}$$

In the notation of Appendix B, we compute from (2.7) the matrix λ to be

$$\begin{aligned} M &= 1 - R^t A\,, \qquad A = c\,1 - (c-1)Q \\ \mathbf{m_1} &= s^2\,(R^t - 1)\,\check{e}_3 \\ \mathbf{m_2} &= \mathbf{0} \\ m_0 &= -2\,s^2 \end{aligned} \tag{4.2}$$

where $R \equiv R(X)$ is given in (C.7) and $Q = \check{e}_3 \check{e}_3{}^t$.

Now from (B.2) we get

$$\lambda^c = \begin{pmatrix} m_0 M^c & 0 \\ -\mathbf{m_1}{}^t M^c & m \end{pmatrix} \tag{4.3}$$

where (c.f. (B.4))

$$\begin{aligned} M^c &= (1 + (c-1)R_{33} - c\,tr R)\,1 + cR + cR^t + c(c-1)\,R^t Q - (c-1)QR \\ m &= -s^2(1 - R_{33}) \end{aligned} \tag{4.4}$$

The next step is to compute the vectors in (2.8). They are given by

$$\begin{aligned} \mathbf{a} &= \mathbf{U} - \frac{1}{2} d\varphi\,\check{e}_3 \\ a_4 &= -d\varphi \\ \mathbf{b} &= A\,\overline{\mathbf{U}} - \frac{1}{2} d\varphi\,\check{e}_3 \\ b_4 &= -(1 + \frac{3}{2}s^2)\,d\varphi - s^2\,\overline{U}_3 \end{aligned} \tag{4.5}$$

On the other hand, the Wess-Zumino action (2.4) results

$$I_{WZ}[g] = \int_\Sigma (dt \wedge *dt - \frac{3}{4} d\varphi \wedge *d\varphi) + I_{WZ}[X] \tag{4.6}$$

With (4.3,6) and after some calculations we get (2.6) in the form

$$W[g] = \int_\Sigma (dt \wedge *dt + \frac{1}{s^2(1 - R_{33})}\,(L_{\varphi\varphi} d\varphi \wedge *d\varphi + L_{XX} - L_{X\varphi})) + i\Gamma[X] \tag{4.7}$$

where ($S \equiv (1 - c\,trR)\,1 + c\,R^t + c^2 R$)

$$
\begin{aligned}
L_{\varphi\varphi} &= \frac{1}{2}(RM^c)_{33} + \frac{1}{4}(1 - R_{33})(1 + 3c^2) \\
L_{XX} &= -s^2(1 - R_{33})\,\mathbf{U}\cdot\wedge * \mathbf{U} + 2\,\mathbf{U}\cdot\wedge(* - i1)M^c A\,\overline{\mathbf{U}} \\
L_{X\varphi} &= \check{e}_3 \cdot S\,\mathbf{U}\wedge(* - i1)d\varphi + \overline{\mathbf{U}}\cdot S\,\check{e}_3 \wedge (* + i1)d\varphi
\end{aligned} \tag{4.8}
$$

and after repeatedly using formulae collected in Appendix C we get

$$
\begin{aligned}
L_{\varphi\varphi} &= -1 + R_{33} + \frac{1}{4}(5 - 3R_{33})(c-1)^2 + \frac{c}{2}(5 - 2R_{33} - trR) \\
L_{XX} &= 2\,(c-1)^2 dx_3 \wedge * dx_3 + 2\,(c+1)^2 dx_0 \wedge * dx_0 \\
&+ \; i2s^2(1 - R_{33})(x_0 dx_3 - x_3 dx_0)\wedge d\theta \\
L_{X\varphi} &= 2\,d\varphi \wedge *(s^2 x_0\,dx_3 - (s+2)^2 x_3\,dx_0) + is^2(1 - R_{33})\,d\theta\wedge d\varphi \\
\Gamma[X] &= -2\int_\Sigma (x_0\,dx_3 - x_3\,dx_0)\wedge d\theta
\end{aligned} \tag{4.9}
$$

From these results we learn two important facts:

- the last term in L_{XX} cancels the Γ contribution;
- the last term in $L_{X\varphi}$ drops out because it gives a total derivative contribution to W;

that lead us to conclude that:

1. the θ variable in X decouples, as should. As we saw in Section 3 this is only a non trivial check of gauge invariance;
2. the three terms that cancel are those that could give rise to the axionic field B, in other words the target obtained is *torsionless*.

This last fact is not expected "a priori". To our kwowledge, a classification of torsionless groups in GWZM is not available. From the model considered here we can argue that the key fact for this to happen lies in the possibility of going to a gauge in which the Wess-Zumino term is zero [8] (which is made explicit in 1.), but a more general argument is lacking.

If the backgrounds are defined as in (2.1), we read from (4.7,9) the non-zero metric components in the (t, φ, x_0, x_3) variables ($\rho \equiv +\sqrt{1 - {x_0}^2 - {x_3}^2}$)

$$
\begin{aligned}
G_{tt} &= 1 \\
G_{\varphi\varphi} &= \frac{c^2}{s^2} + \frac{c-1}{4(c+1)}\frac{{x_0}^2}{\rho^2} + \frac{c+1}{4(c-1)}\frac{{x_3}^2}{\rho^2} \\
G_{00} &= \frac{c+1}{c-1}\frac{1}{\rho^2}
\end{aligned}
$$

[8] In WZM we certainly have zero torsion if $\Gamma = 0$.

$$
\begin{aligned}
G_{33} &= \frac{c-1}{c+1}\frac{1}{\rho^2} \\
G_{0\varphi} &= \frac{c+1}{2(c-1)}\frac{x_3}{\rho^2} \\
G_{3\varphi} &= -\frac{c-1}{2(c+1)}\frac{x_0}{\rho^2}
\end{aligned}
\tag{4.10}
$$

and from (2.9), (B.3) and (4.2,4) the dilaton field

$$
D = \ln(s^4\rho^2) + D_0 \tag{4.11}
$$

We notice here the existence of a manifest isometry, a traslation in the φ variable with Killing vector $K_\varphi = \partial_\varphi$.

If we go back to P variables (3.6) by means of the rotation ($0 \leq R \leq \pi/2$)

$$
x_0 + i\,x_3 = (p_0 + i\,p_3)\,e^{i\frac{\varphi}{2}} = \sin R\,e^{i\psi} = \sin R\,e^{i(\psi_P+\frac{\varphi}{2})} \tag{4.12}
$$

the metric takes the form [9]

$$
\begin{aligned}
G &= dt^2 + \frac{c^2}{s^2}\,d\varphi^2 + \frac{1}{s^2\,\rho^2}\,(\,|c\,e^{i\varphi}+1|^2\,dp_0{}^2 + |c\,e^{i\varphi}-1|^2\,dp_3{}^2 \\
&- 4\,c\sin\varphi\;dp_0\,dp_3\,) \\
&= dt^2 + \frac{c^2}{s^2}\,d\varphi^2 + \frac{1}{s^2}\,(\,|c\,e^{i2\psi}+1|^2\,dR^2 + |c\,e^{i2\psi}-1|^2\,\tan^2 R\,d\psi_P{}^2 \\
&- 4\,c\,\tan R\,\sin 2\psi\;dR\,d\psi_P\,)
\end{aligned}
\tag{4.13}
$$

In this coordinates the metric looks simpler (in particular, has only one non diagonal term), but the isometry is not manifest.

[9] $2\,dx\,dy \equiv dx\otimes dy + dy\otimes dx$.

5 The curvature and the equations of motion.

It is convenient in what follows to introduce an orthonormal basis $\{\omega^a\}$ in the cotangent space of $\mathcal{M}$, $G = \delta_{ab}\,\omega^a \otimes \omega^b$. We choose

$$\begin{aligned}
\omega^1 &= \frac{c}{s}\,d\varphi \\
\omega^2 &= \frac{c+1}{s\,\rho}\,(dx_0 + \frac{x_3}{2}d\varphi) \\
\omega^3 &= \frac{c-1}{s\,\rho}\,(dx_3 - \frac{x_0}{2}d\varphi) \\
\omega^4 &= dt
\end{aligned} \tag{5.1}$$

and its dual in the tangent space ($\omega_a(\omega^b) = \delta_a{}^b$)

$$\begin{aligned}
\omega_1 &= \frac{s}{c}\,(\partial_\varphi\ +\ \frac{x_0}{2}\partial_3\ -\ \frac{x_3}{2}\partial_0) \\
\omega_2 &= \frac{c-1}{s}\rho\,\partial_0 \\
\omega_3 &= \frac{c+1}{s}\rho\,\partial_3 \\
\omega_4 &= \partial_t
\end{aligned} \tag{5.2}$$

From the first Cartan's structure equation (torsionless condition)

$$T^a \equiv d\omega^a + \omega^a{}_b \wedge \omega^b = 0 \tag{5.3}$$

we read the non vanishing connections [10]

$$\begin{aligned}
\omega^1{}_2 &= \omega^3{}_4 = \frac{1}{s}\,\omega^3 \\
\omega^1{}_3 &= -\omega^2{}_4 = \frac{1}{s}\,\omega^2 \\
\omega^2{}_3 &= \frac{c^2+1}{2\,s\,c}\,\omega^1\ +\ \frac{c+1}{s}\frac{x_3}{\rho}\,\omega^2\ -\ \frac{c-1}{s}\frac{x_0}{\rho}\,\omega^3 \\
\omega^1{}_4 &= -\frac{1}{sc}\,\omega^1
\end{aligned} \tag{5.4}$$

By using now the second Cartan's structure equation

$$\Omega^a{}_b \equiv d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2}R^a{}_{bcd}\,\omega^c \wedge \omega^d \tag{5.5}$$

we read the Riemman curvature tensor

$$R_{1212}\ =\ R_{1234} = R_{3434} = \frac{1}{c+1}$$

[10]Remember that in an orthonormal basis the metricity condition $\omega^a{}_b = -\omega^b{}_a$ holds, as well as the general symmetry properties: $R_{abcd} = R_{cdab} = -R_{bacd}$ [11].

$$
\begin{aligned}
R_{1324} &= -R_{1313} = -R_{2424} = \frac{1}{c-1} \\
R_{1223} &= R_{2334} = \frac{2}{c+1}\frac{x_0}{\rho} \\
R_{1323} &= -R_{2324} = -\frac{2}{c-1}\frac{x_3}{\rho} \\
R_{2323} &= \frac{2}{s^2} + R_{22} \\
R_{1423} &= -R_{1414} = \frac{2}{s^2}
\end{aligned}
\tag{5.6}
$$

and contracting, the Ricci tensor $R_{ab} \equiv R_{cacb} = R_{ba}$

$$
\begin{aligned}
R_{11} &= R_{44} = -\frac{4}{s^2} \\
R_{12} &= -R_{34} = -\frac{2}{c-1}\frac{x_3}{\rho} \\
R_{13} &= R_{24} = -\frac{2}{c+1}\frac{x_0}{\rho} \\
R_{14} &= R_{23} = 0 \\
R_{22} &= R_{33} = -2\,(1 + \frac{2}{s^2} + \frac{c-1}{c+1}\frac{{x_0}^2}{\rho^2} + \frac{c+1}{c-1}\frac{{x_3}^2}{\rho^2}\ )
\end{aligned}
\tag{5.7}
$$

Finally, the scalar curvaure $R \equiv {R_a}^a$ is

$$
-\frac{1}{4}\,R = 1 + \frac{4}{s^2} + \frac{c-1}{c+1}\frac{{x_0}^2}{\rho^2} + \frac{c+1}{c-1}\frac{{x_3}^2}{\rho^2} \tag{5.8}
$$

With these results at hand it is straightforward to verify that the graviton-dilaton system given by equations (4.10,11) verify the consistency equations (2.2) with $B = 0$ and $\Lambda = 12$. We do not know if the torsion remains null at higher orders, but we speculate that it is indeed the case. As we anticipate, $t = 0$ and $\rho = 0$ are true singularities of the geometry, where the parametrization (3.7) breaks down.

Here a little disgresion is in order. The value of Λ suggests that the model is conformally invariant at one loop iff $k = \frac{18}{11} \simeq 1.64$. On the other hand, from current algebra arguments [12] the exact central charge of the model is

$$
\begin{aligned}
c_{SU(2,1)/U(2)} &= c_{su(3)} - c_{su(2)} - c_{u(1)} = 8\frac{k}{k-3} - 3\frac{k}{k-2} - 1 \\
&= 4\,+\,6\,\frac{3k-5}{(k-2)(k-3)}
\end{aligned}
\tag{5.9}
$$

Then imposing the cancelation against ghost contribution we obtain the values $k_+ \simeq 3.96$ and $k_- \simeq 1.86$. The second one is near the value obtained perturbatively at first order. It is believed that by taking into account all loop corrections the value of Λ should lead to k_+ or k_-; however k does not seem to be big enough to assert that the perturbative theory necessarily corresponds to k_-. Moreover,

in analogy with the condition that $-k = n$ be a positive integer needed for the quantum consistency of the compact models it is speculated that unitarity would allow only $k > 3$, and if true (the subject is far from being well understood by now) k_+ should be the right value to be considered. We will take $k \in \Re^+$ for which at least the one loop path integral seems to be well defined [17]; see next section for more about. As a last observation, if we consider the "non critical" GWZM, i.e., with a dynamical Liouville field, the allowed values of k are rational: $k_\pm = 4,\ \frac{13}{7}$.

In order to compare with euclidean Einstein gravity, we introduce the metric $G^E \equiv e^D G$. Then the backgrounds (G^E, D) are classical solutions of the action

$$S[G^E, D] = \int_{\mathcal{M}} (*R^E - \frac{1}{2}\,\nabla^E D \wedge *\nabla^E D + *\Lambda e^{-D}) \tag{5.10}$$

which describes a Liouville field coupled to gravity in $d = 4$, and may then be interpreted as a "pseudo-instanton" of this theory. In fact the solution is singular at $t = 0$ and $\rho = 0$ as expected, and the R_{14} and R_{23} components fail to be (anti) self-dual, as usually known instantons are [18]. What is more, it is not asymptotically flat in the usual sense (at least in the standard range of the coordinates of the model that we assume), and gives an infinite value for the action

$$I_{inst} = 12\,\pi^2\,\sinh^4 T\, e^{D_0} \tag{5.11}$$

where T is a cut-off in the t-integration. In the compact coset $SU(3)/U(2)$ the variable t, better to say, its continuation to imaginary values $\tau \equiv i\,t$ is naturally bounded to the interval $[0,\,\pi/2]$, and the action is finite.

A possible interpretation of the solution is as follows. For $t \gg 1$ we have

$$\begin{array}{lcl} G & \to & dt^2 + d\varphi^2 + dR^2 + tg^2 R\, d\psi_P{}^2 \\ D & \to & 4\,t + 2\ln\cos R \\ R & \to & -4\,\sec^2 R \end{array} \tag{5.12}$$

which describes the topology product of a cylinder (a plane in the compact case, for τ near $\pi/2$) and a "trumpet". On the other hand it may be thought as a euclidean coontinuation of the non singular cosmological solution

$$G_{cs} = -dx_0{}^2 + \tanh^2 x_0\, dx_1{}^2 \ +\ dx_2{}^2 \ +\ dx_3{}^2 \tag{5.13}$$

arising from the $SL(2,\Re) \times SO(1,1)^2/SO(1,1)$ model [19]. Then is tempting to interpret the instanton as a path in "euclidean time t" that interpolates two universes, one in a "big bang" phase (singularity at $t = 0$) and other smoothly evolving according to (5.13). We will see in the next section that for finite k very different (and appealing) possibilities arise.

As a final remark we note that being the string coupling constant [4]

$$g_{st} = e^{-D/2}$$

then from (5.11,12) we have

$$I_{inst} = \frac{3\ \pi^2}{4\ {g_{st}}^2}$$

exhibing the usual non perturbative behaviour characterizing the "tunneling amplitud" $exp(-I_{inst})$ for the process described by the instanton.

6 The exact backgrounds

The computation

In references [5,6] an ansatz to obtain the exact metric and dilaton backgrounds was proposed. Here we resume it in a few items.

A) Let X_a be a basis in $\mathcal{G}$ simple and compact, satisfying the algebra

$$[X_a, X_b] = i f_{ab}{}^c X_c \tag{6.1}$$

and $g \in G$. We define left and right currents (that certainly satisfy (6.1)) as linear operators acting on G according to

$$\begin{aligned} \hat{J}^R_a g &\equiv -g X_a \\ \hat{J}^L_a g &\equiv X_a\, g \end{aligned} \tag{6.2}$$

B) Once we read from (6.2) $\hat{J}^{L,R,}_a$ we construct the quadratic Casimir operators in this G-realization,

$$\hat{\Delta}^{L,R}_G \equiv g^{ab} \hat{J}^{L,R}_a \hat{J}^{L,R}_b \tag{6.3}$$

where g^{ab} is the inverse of the Cartan metric $g_{ab} = tr(X_a X_b)$ (for normalizations, see Section 2), and in the same way we construct the Casimir operators $\hat{\Delta}^{L,R}_H$ associated with the subgroup H, by restricting (6.3) to the $\mathcal{H}$ generators. Then we define the Virasoro-Sugawara operators

$$\hat{L}^{L,R}_0 \equiv \frac{1}{k + C_{\mathcal{G}}} \hat{\Delta}^{L,R}_G - \frac{1}{k + C_{\mathcal{H}}} \hat{\Delta}^{L,R}_H \tag{6.4}$$

where $C_{\mathcal{G},\mathcal{H}}$ are the respective dual Coxeter numbers. If $\mathcal{H}$ is semisimple then we will have sums with prefactors corresponding to each simple components [12].

C) We identify the subspace of functions on G dictated by the gauge invariance conditions [11]

$$(\hat{J}^L_a + \hat{J}^R_a) f(g) = 0\ , \quad a = 1, ..., dim\mathcal{H} \tag{6.5}$$

D) Finally we apply the hypothesis of [6]

$$\begin{aligned} (\hat{L}^L_0 + \hat{L}^R_0) f(g) &\equiv -(k + C_{\mathcal{G}})^{-1} \chi^{-1} \partial_\mu (\chi G^{\mu\nu} \partial_\nu) f(g) \\ \chi &= e^D \sqrt{|detG|} \end{aligned} \tag{6.6}$$

from where we can directly get $G^{\mu\nu}$ by looking at the quadratic terms, and a system of first order differential equations to determine χ (and so D) from the linear terms.

Going to our model, we take $X_a \equiv \lambda_a$ the Gell-Mann matrices and consider the parametrization (3.7,8) and (C.4). Let us introduce the commuting linear operators $\hat{\mathbf{X}}$ and $\hat{\mathbf{V}}$

$$\hat{X}_1 = -i\, (x_2\, \partial_3 - x_3\, \partial_2)$$

[11] We remember that $\hat{V}_a = \hat{J}^L_a + \hat{J}^R_a$ are the generators of the vector transformations (A.6).

$$
\begin{aligned}
\hat{X}_2 &= -i\, x_0\, \partial_2 \\
\hat{X}_3 &= -i\, x_0\, \partial_3 \\
V_i &= \frac{i}{2}(v_0\, \partial_i - \epsilon_{ijk}\, v_j\, \partial_k), \quad i = 1,2,3
\end{aligned}
\tag{6.7}
$$

that verify (6.1) with ${f_{ij}}^k = \epsilon_{ijk}$.

Then from (6.2) we read [12]

Right currents

$$
\begin{aligned}
\hat{R}_i &= -R(V)_{ji}\, (\hat{X}_j + u_j(\hat{V}_3 - i\partial_\phi))\ , \quad i = 1,2,3 \\
\mathbf{u} &= \frac{1}{x_2}(x_0\, \check{e}_1 + x_3\, \check{e}_2 - x_2\, \check{e}_3) \\
\hat{R}^+_\alpha &= -\frac{u^{3/2}}{2}(XV)_{1\alpha}\, (\partial_t + i\frac{s}{c}\partial_\varphi) + A^V_0 \partial_\phi + i\, \mathbf{A^V} \cdot \hat{\mathbf{V}} + \mathbf{i}\, \mathbf{A^X} \cdot \hat{\mathbf{X}} \\
i\, 2\, s\, u^{-3/2} A^V_0 &= \frac{c^2+3}{2c}(XV)_{1\alpha} + \frac{cz - z^*}{x_2}\, (XV)_{2\alpha} \\
i\, 2\, s\, u^{-3/2} \mathbf{A^V} &= 2\, (XV)_{2\alpha}\, (-\check{e}_1 + i\check{e}_2\,) + (\frac{s^2}{2c}(XV)_{1\alpha} + \frac{cz - z^*}{x_2}(XV)_{2\alpha})\, \check{e}_3 \\
i\, 2\, s\, u^{-3/2} \mathbf{A^X} &= (c+1)(XV)_{2\alpha}\, \check{e}_1\, - i\, (c-1)(XV)_{2\alpha}\, \check{e}_2\, + \frac{s^2}{2c}(XV)_{1\alpha}\, \check{e}_3 \\
\hat{R}^-_\alpha &= (\hat{R}^+_\alpha)^* \\
\hat{R}_8 &= i\frac{2}{\sqrt{3}}\, \partial_\varphi
\end{aligned}
\tag{6.8}
$$

Left currents

$$
\begin{aligned}
\hat{L}_i &= -\hat{R}_i + 2\, R(V)_{ji} \hat{V}_j\ , \quad i = 1,2,3 \\
\hat{L}^+_\alpha &= \frac{1}{2} V_{1\alpha}(\partial_t - i\frac{s}{c}\partial_\varphi) + A^V_0 \partial_\phi + i\mathbf{A^V} \cdot \hat{\mathbf{V}} + \mathbf{i}\mathbf{A^X} \cdot \hat{\mathbf{X}} \\
i\, 2\, s\, A^V_0 &= -\frac{7c^2 - 3}{2c} V_{1\alpha} + \frac{cz^* - z}{x_2} V_{2\alpha} \\
i\, 2\, s\, \mathbf{A^V} &= c\, V_{2\alpha}\, (\check{e}_1 - i\, \check{e}_2\,) + \ (\frac{s^2}{2c} V_{1\alpha} + \frac{cz^* - z}{x_2} V_{2\alpha})\, \check{e}_3 \\
i\, 2\, s\, \mathbf{A^X} &= -(c+1) V_{2\alpha}\, \check{e}_1 - i(c-1) V_{2\alpha}\, \check{e}_2 + \frac{s^2}{2c}\, V_{1\alpha}\, \check{e}_3 \\
\hat{L}^-_\alpha &= (\hat{L}^+_\alpha)^* \\
\hat{L}_8 &= -\hat{R}_8 + i\, 2\, \sqrt{3}\, \partial_\phi
\end{aligned}
\tag{6.9}
$$

Clearly the first and last equations in (6.9) translate the gauge conditions (6.5) as the independence on ϕ and $\mathbf{v}$, i.e., on the gauge variable V. Restricting us to the gauge invariant subspace, we get the laplacians

$$
\hat{\Delta}^L_G = \hat{\Delta}^R_G = \hat{\Delta}_{U(1)} + \hat{\Delta}_{SU(2)} + \{\hat{R}^+_\alpha, \hat{R}^-_\alpha\}
\tag{6.10}
$$

[12] The index $\alpha = 1, 2$ refers to the combinations $\lambda^\pm_1 = \frac{1}{2}(\lambda_4 \pm i\lambda_5)\ ,\ \ \lambda^\pm_2 = \frac{1}{2}(\lambda_6 \pm i\lambda_7)$.

and acording to (6.4) we have [13]

$$\hat{L}_0^L = \hat{L}_0^R = \frac{1}{k-3}\hat{\Delta}_{SU(2,1)} - \frac{1}{k-2}\hat{\Delta}_{SU(2)} - \frac{1}{k}\hat{\Delta}_{U(1)} \tag{6.11}$$

Carrying out the computations and applying (6.6) we read the inverse metric; the modified basis (5.1) looks

$$\begin{aligned}
\omega^1 &= (\frac{s^2}{c^2} - b)^{-\frac{1}{2}}\, d\varphi \\
\omega^2 &= \frac{c+1}{s\,\rho}\, \beta^{\frac{1}{2}}\, (dx_0 + \frac{f}{1-b\,\frac{c^2}{s^2}}\, \frac{x_3}{2}\, d\varphi - (f-1)\frac{x_3}{x_0}\, dx_3) \\
\omega^3 &= \frac{c-1}{s\,\rho} \left(\frac{f}{1-a\frac{c-1}{c+1}} \right)^{\frac{1}{2}} (dx_3 - (1-b\,\frac{c^2}{s^2})^{-1}\, \frac{x_0}{2}\, d\varphi) \\
\omega^4 &= dt
\end{aligned} \tag{6.12}$$

and after solving the differential equations, the dilaton

$$D = D_0 + \ln \frac{s^3\, c}{|\det\, G|^{\frac{1}{2}}} \tag{6.13}$$

where

$$\begin{aligned}
det\, G &= \frac{\beta\, f}{(1-a\,\frac{c-1}{c+1})(\frac{s^2}{c^2}-b)\,\rho^4} \\
\beta^{-1} &= 1 - \frac{c+1}{c-1}\left(a + (f-1)(\frac{c+1}{c-1} - a)\, \frac{{x_3}^2}{{x_0}^2} \right) \\
f^{-1} &= 1 - \frac{a\,b}{\epsilon}\, (\frac{c+1}{c-1} - a)^{-1}\, \frac{(1-\epsilon)c^2 - 1}{(1-b)c^2 - 1}\, \frac{{x_0}^2}{\rho^2}
\end{aligned} \tag{6.14}$$

and $a = \frac{1}{k-2}$, $b = \frac{4}{k}$, $\epsilon = \frac{2}{k-1}$.

As usual the exact results are not very enlightening and in general the singularity structure becomes highly complicated. Also regions of different signature appears, fact related to the signs in the arguments of the square roots in (6.12), giving rise to bizarre geometries and possible topologies. For example, for $0 < k < 2$ is easy to see that the signature is strictely minkowskian (within the natural range of the group parameters) with φ being the time like coordinate. However some interesting interpretations can be given.

The black plane metrics

Let us consider metrics of the form

$$G_{bp} = -f(x)\, d\tau^2 + f(x)^{-1}\, dx^2 + dy^2 + dz^2 \tag{6.15}$$

[13] The first equality follows from $\Delta^L_G = \Delta^R_G$ and $\Delta^L_H = \Delta^R_H$, this last one valid on gauge invariant functions [5]. Also the usual change $k \to -k$ coming from (2.4) for non compact groups is made [1].

Obviously the topology is $P \times Q$ where P is a plane (or some compactified version of it) and Q an indefined signature submanifold parametrized by (τ, x) coordinates where the geometry is characterized by the function f.

Let us first analyze a "regular" case with

$$f_r(x) = 1 - \frac{\cosh^2 ax_h}{\cosh^2 ax} \tag{6.16}$$

where a, x_h are positive real constants, and introduce the "distorted" coordinate

$$x_* = x \; + \; \frac{1}{2\,a\;\tanh ax_h} \;\ln \frac{\sinh a|x - x_h|}{\sinh a|x + x_h|} \tag{6.17}$$

The inverse relation $x(x_*)$ distinguishes three patches: I for $x > x_h$, II for $|x| < x_h$ and III for $x < -x_h$. By defining null coordinates $u = \tau + x_*$, $v = \tau - x_*$ in regions I and III, and $u = x_* + \tau$, $v = x_* - \tau$ in region II, the metric takes the general form

$$G_{rbp} = -|f_r(x)|\; du\; dv + dy^2 + dz^2 \tag{6.18}$$

The metric is regular in all three patches as can be seen from the scalar curvature (that characterizes all the curvature tensor)

$$R = 2\;a^2\;\cosh^2 ax_h \;\; \frac{-3 + 2\cosh^2 ax}{\cosh^4 ax} \tag{6.19}$$

Then we can glue them as is usually done and the maximally extended conformal Penrose diagram for Q (where each point represents P) so obtained is similar to that of the Kerr solution of general relativity (for $M^2 > a^2, \theta = 0$) [14] with $r_\pm \sim \pm x_h$, and the manifold described by it is geodesically complete. Clearly $x \to \pm\infty$ are asymptotically flat regions, and $x = \pm x_h$ are horizons for observers there (in regions I/III); the geometry is then naturelly interpreted as a "regular black plane" hidden in region II. Its Hawking temperature can be computed by standard methods [4]

$$T_r = \frac{a}{2\pi} \tanh ax_h \tag{6.20}$$

Let us consider now a "singular" case defined by

$$f_s(x) = 1 - \frac{\sinh^2 ax_h}{\sinh^2 ax} \tag{6.21}$$

The distorted coordinate is now defined as in (6.17) with the replacement

$$\tanh ax_h \to (\tanh ax_h)^{-1}$$

[14] See for example figure 27 in page 312 of reference [20].

But now the curvature is

$$R = 2\, a^2\ \sinh^2 ax_h\ \ \frac{3 + 2\sinh^2 ax}{\sinh^4 ax} \tag{6.22}$$

that togheter with (6.21) reveals the existence of flat regions for $|x| \to \infty$, but also displays a true singularity at $x = 0$. Due to this crucial fact we can follow the standard procedure as before and write G_{sbp} as in (6.18), but now we can only glue region I with "half" region II (until the singularity, remember that here x is timelike) because we can not go beyond the singularity where analyticity breaks down; similar remarks are made for regions III and the other half of region II, which are "parity" reflected patches of the first ones. The maximally extended conformal Penrose diagram is then similar to Schwarzchild's. We can say that the singularity at $x = 0$ separates two worlds; we can not certainly pass through the singular black plane, and once we go across the horizon at x_h we will die there after finite proper time. The Hawking temperature for this "singular black plane" is

$$T_s = \frac{a}{2\pi} \coth ax_h \tag{6.23}$$

Now let us establish what these geometries has to do with us. Let us consider the general case of finite $k \neq 2, 3, 4$. Then it is not difficult to show that exists $0 < t_k < \infty$ such that the exact solution given by (6.12,13) has the limit

$$G \ \stackrel{t \gg t_k}{\longrightarrow}\ dt^2 + \frac{k}{k-4}\, d\varphi^2 + \frac{k-2}{k-3}\ \frac{dr^2}{1-r^2} + \frac{k-4}{k-3}\ \frac{r^2}{\frac{k-4}{k-2} - r^2}\, d{\psi'_P}^2 \tag{6.24}$$

$$D \ \stackrel{t \gg t_k}{\longrightarrow}\ 4\, t + \frac{1}{2} \ln |(1-r^2)(\frac{k-4}{k-2} - r^2)| \tag{6.25}$$

where polar coordinates $(r \equiv \sin R, \psi)$ as in (4.12) has been introduced and

$$\psi'_P = \psi - \frac{k}{k-4}\ \frac{\varphi}{2}$$

Now let us take $0 < k < 2$ (for example, the conformal value $k = k_-$ discussed after (5.9)). Then by making the change of variables

$$r^2 = 1 - \frac{2}{2-k}\ \sinh^2 ax \tag{6.26}$$

with $a = \frac{1}{\sqrt{|k-2|}}$, it is easy to show that the *line element*

$$ds^2 = (k-3)\, G$$

tends to the the regular black plane metric with the further identifications

$$y \ \ = \ \ i\, \sqrt{|3-k|}\ \ t$$

$$
\begin{aligned}
z &= \sqrt{k\,\frac{|3-k|}{|4-k|}}\ \varphi \\
\tau &= i\,\sqrt{|4-k|}\ \psi'_P
\end{aligned}
\tag{6.27}
$$

and x_h defined by $\sinh^2 ax_h = 1 - k/2$.

On the other hand, in the case $4 < k < \infty$ the change of variables

$$
r^2 = 1 - \frac{2}{k-2}\ \cosh^2 ax \tag{6.28}
$$

leads to

$$
ds^2 \ \stackrel{t \gg t_k}{\longrightarrow}\ -G_{sbp} \tag{6.29}
$$

with a as before, $\sinh^2 ax_h = -2 + k/2$, and the identifications are (6.27) with the replacement $z \to iz$. The dilaton field in both cases is given by

$$
D \ \stackrel{t \gg t_k}{\longrightarrow}\ 4\,t + \ln \sinh 2a|x| \tag{6.30}
$$

From these results we are in conditions of interpreting the *exact* solutions (6.12), as we made in the $k = \infty$ case, as some kind of instantons that "tunnel" from $t \to 0$ highly singular universes (whose expressions being little ilumining we do not write) to static black plane like universes for $t \gg t_k$. We also notice from (6.30) that $t \gg t_k$ is a weak coupling phase except near the black plane $x \to 0$ where we go to an strong coupling region.

Let us finally remark that the χ field introduced in (6.6) results k-independent, as verified for some models in [5]. This result gives further strong support to the non renormalization theorem conjectured there for any GWZM from path integral measure conformal invariance arguments.

7 The dual backgrounds

In reference [21] was showed that it is possible to obtain another solution to the one loop equations (2.2) starting from one which has an isometry. Explicitely, if (G, B, D) are backgrounds satisfying (2.2) that in some coordinate system are independent of the coordinate φ, then

$$
\begin{aligned}
\tilde{G}_{\varphi\varphi} &= {G_{\varphi\varphi}}^{-1} \\
\tilde{G}_{\varphi\alpha} &= \frac{B_{\varphi\alpha}}{G_{\varphi\varphi}} \\
\tilde{G}_{\alpha\beta} &= G_{\alpha\beta} - \frac{1}{G_{\varphi\varphi}}(G_{\varphi\alpha}\, G_{\varphi\beta} \; - \; B_{\varphi\alpha}\, B_{\varphi\beta}) \\
\tilde{B}_{\varphi\alpha} &= \frac{G_{\varphi\alpha}}{G_{\varphi\varphi}} \\
\tilde{B}_{\alpha\beta} &= B_{\alpha\beta} + \frac{1}{G_{\varphi\varphi}}(G_{\varphi\alpha}\, B_{\varphi\beta} \; - \; B_{\varphi\alpha}\, G_{\varphi\beta}) \\
\tilde{D} &= D + \ln|G_{\varphi\varphi}|
\end{aligned}
\tag{7.1}
$$

where $\alpha, \beta \neq \varphi$, is also a solution. The existence of it is sometimes referred as "target space duality" or "abelian duality". As we saw in Section 4, (4.10,11) fulfills the requirements and then a dual solution may be straightforwardly obtained from (7.1). For sake of completeness we present it,

$$
\begin{aligned}
\tilde{G} &= dt^2 \; + \; \frac{1}{G_{\varphi\varphi}}\, d\varphi^2 + \frac{1}{4\; G_{\varphi\varphi}\; \rho^2} \; (\; (\frac{4\; c^2}{(c-1)^2} + \frac{{x_0}^2}{\rho^2})\; d{x_0}^2 + 2\; \frac{x_0\, x_3}{\rho^2} \; dx_0\, dx_3 \\
&+ \; (\frac{4\; c^2}{(c+1)^2} + \frac{{x_3}^2}{\rho^2}\,)\; d{x_3}^2 \;) \\
\tilde{B} &= \frac{1}{2\; G_{\varphi\varphi}\; \rho^2} d\varphi \wedge (\frac{c+1}{c-1}\; x_3\; dx_0 \; - \; \frac{c-1}{c+1}\; x_0\; dx_3) \\
\tilde{D} &= \tilde{D}_0 + \ln|s^4\; \rho^2\; G_{\varphi\varphi}|
\end{aligned}
\tag{7.2}
$$

We notice that the crossing terms in (4.3) does not appear in (7.2), at expenses of the axionic field. Also the metric present a submetric in the (t, x_0, x_3) variables; formally the Cotton-Darboux theorem [20] assures us that it is possible to diagonalize it but unfortunately we have not succeeded in doing it.

In [22] was showed that if the coordinate φ is periodic, then both solutions are equivalent, i.e., they describe the same conformal theory. In the natural range of our parameters, φ is in fact periodic, and then both (4.10,11) and (7.2) should be equivalent. This can be understood from the GWZM point of view by noting that, having gauged a subgroup with a semisimple algebra containing a $u(1)$ subalgebra, there exists the possibility of considering other model by *axial* gauging the $u(1)$ (see footnote 2). We then conclude that the one loop backgrounds (7.2) are those of the $SU(2,1)/SU(2)_{vector} \times U(1)_{axial}$ GWZM.

8 Conclusions

We have presented in this paper an study of the possible effective geometries underlying a coset model based on the pseudo-unitary group $SU(2,1)$, to our knowledge the first one that considers $SU(p,q)$ groups with $p+q>2$.

In the natural range of the parameters the one loop metric is strictly positive definite and so it does not present "horizons", but is singular on two dimensional manifolds $t=0$ (disk) and $\rho=0$ ($\Re^2$). It may be possible that by changing the topology (e.g., limiting the range of coordinates or compactifying some dimensions) a "regular" gravitational instanton may be obtained. For example, if we introduce in (4.13) the x variable by

$$\sin R = e^{-x+t^\nu} \ , \quad 0<\nu<1 \tag{8.1}$$

then we have for $t \gg 1$,

$$G \to dt^2 + d\varphi^2 - dx^2 - d\psi_P{}^2 \tag{8.2}$$

that is, G results asymptotically flat on $\Re^2 \times T^2$ (the Riemann tensor in fact vanishes). Anyway it does not seem any such modified theory will be fully represented by an exact conformal field theory, because only some patch would be covered by the GWZM considered here.

For finite k (the physical case) the picture drastically changes. Regions of different signature appears, and the structure of the singularities becomes highly complicated. In the examples considered we remain with them, differing from the $2-d$ black hole model where a possible mechanism to evite the singularity seems to work [23].

A question non addressed in this paper is the global topology of the exact target manifold; we have in fact loosely ignored the ranges of the coordinates in the discussions of section 6, although is clear that (6.12,13) is presumibly a solution of the (unknown) exact background field equations independently of them. In our opinion only the study of the quantum theory of the model and possible consistency conditions (e.g., identification of field operators with current algebra primary fields, renormalization, unitarity, ecc.) needed for its existence can give light on the problem.

Finally we remark that, as it occurs with other string solutions, the existence of event horizons with topology different from S^2 (in our case, a plane) is not in contradiction with Hawking's theorem, because our solution has $\Lambda = 12 > 0$ that gives a negative Liouville potential in (5.10) which violates the dominant energy condition [4].

Appendix

A U(p,q) parametrization

Let g an arbitrary element of $\mathbb{C}^{(p+q)\times(p+q)}$,

$$g = \begin{pmatrix} A & B' \\ C'^{\dagger} & D \end{pmatrix} \tag{A.1}$$

where A is $p\times p$, D is $q\times q$ and B', C' are $p\times q$ complex matrices. Let η the diagonal element given by $A=1$, $D=-1$ and $B'=C'=0$. Then the condition $g\eta g^{\dagger}=\eta$ define the elements of $U(p,q)$, and leads to the set of equations

$$\begin{aligned} AA^{\dagger} &= 1 + B'B'^{\dagger} \\ DD^{\dagger} &= 1 + C'^{\dagger}C' \\ AC' &= B'D^{\dagger} \end{aligned} \tag{A.2}$$

The first two equations are solved respectively by [15]

$$\begin{aligned} A &= (1+B'B'^{\dagger})^{\frac{1}{2}}\, U \\ D &= (1+C'^{\dagger}C')^{\frac{1}{2}}\, V \end{aligned} \tag{A.3}$$

where $U\in U(p)$ and $V\in U(q)$ are arbitrary. If we reparametrize: $C' = U^{\dagger}C$, $B'=BV$, last equation in (A.2) is solved by $B=C$, and then

$$\begin{aligned} g(C,U,V) &= T(C)\ H(U,V) \\ T(C) &= \begin{pmatrix} (1+CC^{\dagger})^{\frac{1}{2}} & C \\ C^{\dagger} & (1+C^{\dagger}C)^{\frac{1}{2}} \end{pmatrix} \in U(p,q)/U(p)\otimes U(q) \\ H(U,V) &= \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \in U(p)\otimes U(q) \end{aligned} \tag{A.4}$$

By making the change of variables $C=(NN^{\dagger})^{-\frac{1}{2}}\sinh(NN^{\dagger})^{\frac{1}{2}}N$ we can write the coset element as [16]

$$T(C) = exp\begin{pmatrix} 0 & N \\ N^{\dagger} & 0 \end{pmatrix} \tag{A.5}$$

Let us remark that analogous coset decompositions can be considered in terms of non compact versions of the maximal compact subgroup $U(p)\otimes U(q)$. Also they lead to the corresponding ones to the group $O(p,q)$ by taking the apropiate real sections.

Under an adjoint transformation $g^h = hgh^{\dagger}$ with $h\equiv H(h_1,h_2^{\dagger})\in U(p)\otimes U(q)$, g transforms as:

$$C^h = h_1\ C\ h_2$$

[15]For any complex M the matrix $MM^{\dagger}$ is certainly hermitic and non-negative, and then arbitrary powers of it are well defined through its diagonal form.

[16]For an extensive treatment of coset spaces, see [24].

$$\begin{aligned} U^h &= h_1\, U\, h_1^\dagger \\ V^h &= h_2\, V\, h_2^\dagger \end{aligned} \tag{A.6}$$

For $q = 1$, C is a p-dimensional complex vector and V a phase. By restricting ourselves to $SU(p,1)$ we have $u \equiv \det U = V^\dagger$; clearly the topology of $SU(p,1)$ is $\Re^{2p} \times U(p)$ and its maximal compact subgroup is $U(p)$. A coset decomposition of $U(p)$ wrt its invariant subgroup $SU(p)$ yields, for $p = 2$, the parametrization used in the text (c.f. (3.3)), $U(2)$ being generated by $\{\lambda_1, \lambda_2, \lambda_3, \lambda_8\}$ and the argument of T in (A.5) by the other Gell-Mann matrices.

B Some relations for 4×4 matrices

Here we collect some useful formulae for computing λ^{-1} in Section 3.

Given an arbitrary 4×4 matrix in the form

$$\lambda = \begin{pmatrix} M & \mathbf{m_1} \\ \mathbf{m_2}^{\mathbf{t}} & m_0 \end{pmatrix} \tag{B.1}$$

where M is a 3×3 matrix, $\mathbf{m_1}, \mathbf{m_2}$ are 3-vectors and m_0 a number, then direct inspection shows that its cofactor matrix is given by

$$\lambda^c = \begin{pmatrix} \tilde{M} & -M^c\mathbf{m_2} \\ -\mathbf{m_1}^{\mathbf{t}}\mathbf{M^c} & m \end{pmatrix}$$

$$\tilde{M} = m_0 M^c - (M^t - tr M\, 1)(\mathbf{m_2 m_1}^{\mathbf{t}} - \mathbf{m_1}^{\mathbf{t}}\mathbf{m_2}\ \mathbf{1}) - (\mathbf{m_2 m_1}^{\mathbf{t}}\mathbf{M^t} - \mathbf{m_2}^{\mathbf{t}}\mathbf{M m_1}\ \mathbf{1}) \tag{B.2}$$

and its determinant by

$$l \equiv \det \lambda = m_0\, m - \mathbf{m_1}^{\mathbf{t}}\mathbf{M^c}\mathbf{m_2} \tag{B.3}$$

In these equations $m \equiv \det M$, and

$$\begin{aligned} M^c &= (M^2)^t - tr M M^t + tr M^c 1 \\ 2\, tr M^c &= (tr M)^2 - tr M^2 \end{aligned} \tag{B.4}$$

For $M \equiv 1 - A$ we have

$$\begin{aligned} m &= 1 - a + tr(A^c - A), \quad a \equiv \det A \\ M^c &= (1 - tr A)\, 1 + A^t + A^c \end{aligned} \tag{B.5}$$

Finally, from (B.5) for $M \equiv R \in O(3)$ we have the useful relation

$$R^2 - tr R\, R = R^t - tr R\, 1 \tag{B.6}$$

C $SU(2)$ miscelaneous.

An arbitrary matrix $X \in SU(2)$ can be written as

$$X = x_0\, 1 + i\, \mathbf{x} \cdot \sigma = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} \tag{C.1}$$

where σ are the Pauli matrices, $tr(\sigma_i \sigma_j) = 2\, \delta_{ij}$, and

$$\begin{aligned} z &= x_0 + i x_3\,, \quad w = x_2 + i x_1 \\ 1 &= {x_0}^2 + \mathbf{x} \cdot \mathbf{x} = z z^* + w w^* \end{aligned} \tag{C.2}$$

If $w = \rho\, e^{i\theta}\,,\ 0 < \rho < 1\,,$ we have

$$\begin{aligned} X &= e^{i \frac{\theta}{2} \sigma_3}\, \overline{X}\, e^{-i \frac{\theta}{2} \sigma_3} \\ \overline{X} &= \begin{pmatrix} z & \rho \\ -\rho & z^* \end{pmatrix} \end{aligned} \tag{C.3}$$

The Maurer-Cartan form associated with X is defined by

$$\begin{aligned} \omega(X) &\equiv X^{-1} dX = i\, \mathbf{U}(\mathbf{x}) \cdot \sigma \\ U_i(x) &= x_0\, dx_i - x_i\, dx_0 + \epsilon_{ijk}\, x_j\, dx_k \end{aligned} \tag{C.4}$$

and analoguously $\overline{\omega}(X) \equiv dX X^{-1} = i\, \overline{\mathbf{U}}(\mathbf{x}) \cdot \sigma$, with $\overline{\mathbf{U}}(\mathbf{x}) = -\mathbf{U}(-\mathbf{x})$.

The adjoint representation matrix of X,

$$R(X)_{ij} \equiv \frac{1}{2} tr(\sigma_i X \sigma_j X^\dagger) \tag{C.5}$$

is given by

$$R(X)_{ij} = (2\, x_0^2 - 1) \delta_{ij} + 2\, (x_i\, x_j + x_0\, \epsilon_{ijk}\, x_x) \tag{C.6}$$

Particularly useful in the text are the variables

$$\begin{aligned} R_{33} &= 1 - 2\, ({x_1}^2 + {x_2}^2) = -1 + 2({x_0}^2 + {x_3}^2) \\ trR &= 4{x_0}^2 - 1 \end{aligned} \tag{C.7}$$

From these formulae the following expressions are obtained:

$$
\begin{aligned}
\mathbf{U}\cdot\wedge *\,\mathbf{U} &= dx_0\wedge *dx_0 + d\mathbf{x}\cdot\wedge *\,\mathbf{dx}\\
2\,\mathbf{U}\cdot\wedge *\,\overline{\mathbf{U}} &= (trR+3)dx_0\wedge *dx_0 + (trR-1)d\mathbf{x}\cdot\wedge *\,\mathbf{dx}\\
\mathbf{U}\cdot\wedge\overline{\mathbf{U}} &= 2\,x_0\,\epsilon_{ijk}\,x_i\,dx_j\wedge dx_k\\
U_3\wedge\overline{U}_3 &= (1-R_{33})\,(x_0\;dx_3 - x_3\;dx_0)\wedge d\theta\\
\mathbf{e_3}\cdot\mathbf{R}\,\overline{\mathbf{U}} &= -2\,d(x_0x_3) + trR\;(x_0\;dx_3 - x_3\;dx_0 + \frac{1}{2}(1-R_{33})\;d\theta)\\
\mathbf{e_3}\cdot\mathbf{R^t}\,\mathbf{U} &= -2\,d(x_0x_3) + trR\;(x_0\;dx_3 - x_3\;dx_0 - \frac{1}{2}(1-R_{33})\;d\theta)\qquad\text{(C.8)}
\end{aligned}
$$

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